INTEGRAL SIDES OF RIGHT TRIANGLES.

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$$a^2+b^2=c^2.$$

Problem I. To find integral sides of right triangles.

Rule 1. Take two integers, both odd or both even. $\frac{1}{2}$ the sum of their squares equals the hypotenuse, or c; $\frac{1}{2}$ the difference of their squares equals one of the legs, or b; and their product equals the other leg, or a.

Rule 2. Take any two integers. The sum of their squares equals the hypotenuse, or c; the difference of their squares equals one of the legs, or b; and twice their product equals the other leg, or a.

Rule 3. If *prime* integral sides are desired, the integers chosen must be prime to each other; in Rule 1, both odd; and in Rule 2, one odd and the other even.

Note. Rules 1 and 2 hold good also for fractional values. These rules are deduced from the two formulas mentioned in Problem II, and, to avoid repetition, are not discussed in this problem.

Problem II. Given one of the legs of a right triangle of integral sides to find the other leg and the hypotenuse.

The sides of a right triangle depend upon the equation $a^2+b^2=c^2$, in which a and b are the legs and c the hypotenuse of the triangle.

In the discussion of this problem, a is taken as the *given leg.*

When integral equations of the form $a^2+b^2=c^2$ are considered, the sets of values for a, b. and c are divided into two classes: (1) Those having no common factor; a, b, and c being prime integral values. (2) Those having a common factor; a, b, and c being found by multiplying a, b, and c of the first class by the highest common factor.

Sets of *prime* integral values are, therefore, the basis of work.

In right triangles of integral sides, any integer from 3 up may be taken as the value of one of the legs.

There are three kinds of integers to be considered: (1) Odd numbers; (2) Even numbers divisible by 4; and (3) Even numbers that are 2 times an odd number.

a may, then, be any one of these three kinds of numbers.

When a is *an odd number*, we have the formula

$$(mn)^2+\left(\frac{m^2-n^2}{2}\right)^2=\left(\frac{m^2+n^2}{2}\right)^2$$

by means of which to find b and c, so that a, b, and c have no common factor.

$$mn=a,\ \frac{m^2-n^2}{2}=b, \text{ and } \frac{m^2+n^2}{2}=c.$$

m and n are odd and are prime to each other, and $m>n$. There are as many

sets of prime integral values of a, b, and c as m and n can be made sets of odd, prime, integral factors, the product of each set of which factors equals a.

When a is *an even number divisible by* 4, we have the formula $(2mn)^2+(m^2-n^2)^2=(m^2+n^2)^2$, by means of which to find b and c, so that a, b, and c have no common factor. $2mn=a$, $m^2-n^2=b$, and $m^2+n^2=c$. m and n are prime to each other, one being odd, the other even; and $m>n$. There are as many sets of prime integral values of a, b, and c as m and n can be made sets of prime integral factors, the product of each set of which factors equals $\frac{1}{2}a$.

When a is *an even number that is* 2 *times an odd number*, we first find the set or sets of values for a equal to the odd number, and then multiply them by 2.

When a contains odd factors other than itself and unity, or even factors divisible by 4, there are other sets of values, in which a, b, and c have a common factor. There are as many sets of values of this kind as the sets of prime integral values that can be found for the odd factors and the even factors divisible by 4, cantained in a. In this case we first find the sets of prime integral values for each of the factors and then multiply them by the respective numbers that produce a.

In problems relating to the integral sides of right triangles, unity and the number itself are considered factors of a number.

For the purpose of bringing out the foregoing statements more clearly to the mind of the reader, we shall present them by way of illustration.

Put $a=3$, the lowest integer for integral sides of right triangles. Then $mn=3=3\times1$; whence $m=3$, $n=1$. Substituting these values in the formula for $a=$an odd number, we find $b=\frac{1}{2}(3^2-1^2)=4$, and $c=\frac{1}{2}(3^2+1^2)=5$. There is but one set of values; viz., 3, 4, 5.

Put $a=4$. Then $2mn=4=2\times2\times1$; whence $m=2$, $n=1$. Substituting these values in the formula for $a=$an even number divisible by 4, we find $b=2^2-1^2=3$, and $c=2^2+1^2=5$. This set of values, 4, 3, 5, is the same as that for $a=3$, only a and b have interchanged values. There is but one set.

Put $a=12$. Then $2mn=12=2\times6\times1$ and $2\times3\times2$. There are, therefore, two sets of prime integral values. To find first set, $m=6$, $n=1$. To find second set, $m=3$, $n=2$. Whence the sets are 12, 35, 37; and 12, 5, 13. But $12=4\times3$ and 3×4. Hence there are two other sets of values, each set having a common factor. When $a=3$, $b=4$, $c=5$. When $a=4$, $b=3$, $c=5$. Multiplying these sets by the respective numbers that produce $a=12$, we obtain the required sets, 12, 16, 20; and 12, 9, 15, making in all 4 sets.

Put $a=15$. Then $mn=15=15\times1$ and 5×3. There are, therefore, two sets of prime integral values: 15, 112, 113; and 15, 8, 17. But as $15=5\times3$ and 3×5, there are also two sets of values, each set having a common factor. When $a=3$, $b=4$, $c=5$. When $a=5$, $b=12$, $c=13$. Whence the required sets are 15, 20, 25; and 15, 60, 65,—in all 4 sets.

In order to find the number of sets of values that can be formed for $a=$an integer, we shall illustrate by taking $a=60$. Then $2mn=60=2\times30\times1$, $2\times15\times2$, $2\times10\times3$, and $2\times6\times5$. Hence there are 4 sets of prime integral val-

ues. But 60 contains also the following factors that are odd numbers: $3=3\times1$; $5=5\times1$; and $15=15\times1$ and 5×3. These give 4 more sets. The factors that are even numbers divisible by 4, are $4=2\times2\times1$; $12=2\times6\times1$ and $2\times3\times2$; and $20=2\times10\times1$ and $2\times5\times2$. These give 5 additional sets. Hence for $a=60$, there are 13 sets of values for integral sides of right triangles.

A THEOREM ON PRISMOID.

By P. H. PHILBRICK, C. E., Pineville, Louisiana.

THEOREM. *To prove that the error of the "end area volume" of any prismoid or solid to which the prismoidal formula applies, is twice the error of the "middle area volume" and on the opposite side of the true result.*

Let A and B represent the end areas, M the middle area, and l the length of the prismoid.

Then the true volume is, $V=\frac{1}{6}l(A+4M+B)$.......................(1),

the end area volume is, $V_e=\frac{1}{2}l(A+B)$.............. ....(2),

and the middle area volume is, $V_m=lM$...................................(3).

Now (1)—(2) gives error of $(2)=V-V_e=\frac{1}{6}l(4M-2A-2B)$ ....(4),

and (1)—(3) gives error of $(3)=V-V_m=\frac{1}{6}l(A+B-2M)$..................(5).

But (4) is twice (5) with a contrary sign.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

74. Proposed by JOHN T. FAIRCHILD, Principal of Crawfis College, Crawfis College, Ohio.

When U. S. bonds are quoted in London at $108\frac{3}{4}$ and in Philadelphia at $112\frac{1}{2}$, exchange \$4.89$\frac{1}{2}$, gold quoted at 107, how much more was a \$1000 U. S. bond worth in London than in Philadelphia?

No solution of this problem has been received.